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Ergodicity conditions for a continuous one-dimensional loss network

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Abstract. One dimensional continuous loss networks are spatial birth-and-death processes which can be dominated by a multitype branching process. Using the Peron-Frobenius theory for sub-criticality of branching process we obtain a sufficient condition for ergodicity of one-dimensional loss networks on $\mathbb{R}$ with arbitrary length distribution π and cable capacity C.

Keywords: clan of ancestors, branching process, Peron-Frobenius root.

Mathematical subject classification: Primary: 60J80, 60G55; Secondary: 15A18.

1 Introduction

The continuous unbounded one dimensional loss network is a spatial birth and death process and can be interpreted as a system where users are arranged along an infinitely long cable and a call between two points on the cable $s_1, s_2 \in \mathbb{R}$ involves just that section of the cable between s_1 and s_2. Past any point along its length the cable has the capacity to carry simultaneously up to C calls: a call attempt between $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$, is lost if past any point of the interval $[s_1, s_2]$ the cable is already carrying C calls. Suppose that calls are attempted at points in $\mathbb{R}$ following a homogeneous Poisson process with rate λ. Assume that the section of the cable demanded by a call has distribution π with finite mean ρ_1 and the duration of a call has exponential distribution with mean one. Assume that the location of a call, the cable section needed and its duration are independent. Let $m(s,t)$ be the number of calls in progress past point s on the cable at time t. Kelly (1991) conjectured that $((m(s,t), s \in \mathbb{R}), t \geq 0)$ has a unique invariant measure, given by a stationary $M/G/\infty$ queue (Markov arrivals, general service time and infinite servers) conditioned to have at most C

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clients at all times. Ferrari and Garcia (1998) used a continuous (non-oriented) percolation argument to prove the above conjecture whenever π has finite third moment and the arrival rate λ is sufficiently small. Fernández, Ferrari and Garcia (2002) using an oriented percolation argument and a domination by a branching process improved this bound to

$$\lambda(\rho_2 + \rho_1 + 1) < 1 \tag{1.1}$$

where ρ_1 and ρ_2 are the first and second moment of distribution π respectively. Their argument is more general and it is based on a graphical representation of spatial birth and death processes and it is the basis for the perfect simulation scheme "*Backward-Forward Algorithm*", described in Fernández, Ferrari and Garcia (2002). In this work, using the same graphical representation and the Peron-Frobenius theory for sub-criticality of branching process for the specific case of loss networks we obtain a new bound given by

$$\lambda(\sqrt{\rho_2} + \rho_1) < 1. \tag{1.2}$$

2 Spatial loss networks

Let $\mathcal{G}$ be a family of objects γ ($\gamma = (x, u)$, $x \in \mathbb{R}$, $u \in \mathbb{R}_+$) which we will call *individuals* or *calls* and consider a state space $\mathcal{S} = \{\xi \in \mathbb{N}^{\mathcal{G}} : \xi(\gamma) \neq 0$ only for a countable set of $\gamma \in \mathcal{G}\}$.

We first introduce the *free process* which is a birth-and-death process characterized by the fact that its birth rate does not depend on the actual configuration of the system, that is there exists a Radon measure ν on $\mathcal{G}$ such that the calls are born at intensity ν and last for a random time exponentially distributed with mean one. The generator of the free process is:

$$\begin{aligned} A^0 F(\eta) = \int_{\mathcal{G}} \nu(d\gamma)\, [F(\eta + \delta_\gamma) - F(\eta)] \\ + \sum_{\gamma \in \mathcal{G}:\ \eta(\gamma) > 0} \eta(\gamma)\, [F(\eta - \delta_\gamma) - F(\eta)] \end{aligned} \tag{2.1}$$

Here δ_γ is the configuration with only one point at γ and $(\eta+\xi)(\theta) = \eta(\theta)+\xi(\theta)$ (coordinatewise sum). This free process always exists and is ergodic whichever the choice of w. In the particular case where $\omega(\gamma) = \lambda$ the invariant measure is the λ-homogeneous Poisson process.

For the loss network, its generator can be written (Fernández, Ferrari and Garcia, 2002) as

$$AF(\eta) = \int_{\mathcal{G}} \bar{\nu}(d\gamma)\, M(\gamma,\eta)\,[F(\eta+\delta_\gamma) - F(\eta)] \\ + \sum_{\gamma\in\mathcal{G}\,:\,\eta(\gamma)>0} \eta(\gamma)\,[F(\eta-\delta_\gamma) - F(\eta)] \tag{2.2}$$

where

$$\bar{\nu}(d(x,u)) \;=\; \lambda\,\pi(du)\,dx\ , \tag{2.3}$$

and denoting $\eta(y) = \sum_{(x,u)\in\eta} \mathbf{1}\{y \in (x, x+u)\}$, the number of calls of η that uses point y,

$$M((x,u),\eta) = \mathbf{1}\{(\eta+\delta_{(x,u)})(y) \le C, \forall u \in \mathbb{R}\}. \tag{2.4}$$

The factor $\bar{\nu}$ represents a basic birth-rate density due to an "internal" Poissonian clock and the factor M acts as an unnormalized probability for the individual to be actually born once the internal clock has rang. The birth is hindered or reinforced according to the configuration η.

We introduce a function $I : \mathbf{G}\times\mathbf{G} \to \{0,1\}$

$$I(\gamma,\theta) = \mathbf{1}\{sup_\eta\{|M(\gamma,\eta) - M(\gamma,\eta+\delta_\theta)\}| > 0\}. \tag{2.5}$$

where $\delta_\theta(\gamma) = \mathbf{1}\{\gamma = \theta\}$ is the configuration having unique individual θ and the supremum is taken over the set of all configurations ξ such that ξ and $\xi+\delta_\theta$ are in the set of allowed configurations (either $\{0,1\}^{\mathbf{G}}$ or $\mathbb{N}^{\mathbf{G}}$). The function $I(\gamma,\theta)$ indicates which individuals θ may have an influence in the birth-rate of the individual γ, that is if $I(\gamma,\theta)$ =1, the presence (or absence) of θ modifies the birth rate of γ and then we say that θ is *incompatible* to γ.

The above arguments induces the graphical representation of the birth and death process which is the basis for the perfect simulation scheme "*Backward-Forward Algorithm*", described in Fernández, Ferrari and Garcia (2002). This algorithm involves the "thinning" of a marked Poisson process –the *free process*– which dominates the birth-and-death process, and it involves a time-backward and a time-forward sweep. The initial stage of the construction is done *toward the past*, starting with a finite window and retrospectively looking to *ancestors*, namely to those births in the past that could have (had) an influence on the current birth. The construction of the *clan of ancestors* constitutes the time-backward sweep of the algorithm. Once this clan is completely constructed, the algorithm proceeds in a time-forward fashion "cleaning up" successive generations according to appropriate penalization schemes. The relation "being ancestor of" induces a backward in time *contact/oriented percolation* process. The algorithm is applicable as long as this oriented percolation process is sub-critical.

Let $N = \{ (\xi_1, T_1), (\xi_2, T_2), ...\}$ be a homogeneous Poisson Process with rate λ in $\mathbb{R} \times [0, \infty)$, and let $S_1, S_2, ...$ be i.i.d. random variables exponentially distributed with mean one and let $U_1, U_2, ...$ be i.i.d. random variables with common distribution π. Assume the family of variables $\{S_1, S_2, \ldots\}$, $\{U_1, U_2, \ldots\}$ and the Poisson process are all independent. Consider the random rectangles

$$R_i = \{(x, y); \xi_i \le x \le \xi_i + U_i, T_i \le y \le T_i + S_i\}.$$

Then $\{R_i, i \ge 1\} = \{(\xi_i, T_i) + D_i, i \ge 1\}$ is a Boolean model in $\mathbb{R}^2$ where $D_i = [0, U_i] \times [0, S_i]$ and represents the free process of calls.

Now, for each rectangle R_i we associate an independent mark $Z_i \sim U(0, 1)$, and each marked rectangle we identify with the marked point $(\xi_i, T_i, S_i, U_i, Z_i)$. We recognize in the marked point process

$$\mathbf{R} = \{(\xi_i, T_i, S_i, U_i, Z_i),\ i = 1, 2, \ldots\}$$

a graphical representation of the birth and death process with constant birth rate λ, and constant death rate, equal to 1. We call this free process α and Z_i will serve as a flag of allowed births. Calling $R = (\xi, \tau, s, u, z)$, we use the notation

$$\text{Basis(R)}= (\xi, \xi + u), \quad \text{Birth(R)}= \tau, \quad \text{Life(R)}= [\tau, \tau\text{+s}], \quad \text{Flag(R)}= \text{z}.$$

We need a series of definitions:

- For an arbitrary point $(x, t) \in \mathbb{R}^2$ and a rectangle R define as their ancestor sets

$$A_1^{(x,t)} = \{R \in \mathbf{R} |\ x \in \text{Basis}(R), t \in \text{Life}(R)\} \tag{2.6}$$

$$A_1^R = \{R' \in \mathbf{R} |\ \text{Birth}(R') \le \text{Birth}(R),\ R' \cap R \neq \emptyset\} \tag{2.7}$$

- Define recursively the generations ($n > 1$) of the above sets that is, the nth generation of ancestors:

$$A_n^{(x,t)} = \{R'' | R'' \in A_1^{R'} \text{ for some } R' \in A_{n-1}^{(x,t)}\} \tag{2.8}$$

$$A_n^R = \{R'' | R'' \in A_1^{R'} \text{ for some } R' \in A_{n-1}^R\} \tag{2.9}$$

We say that there is *backward oriented percolation* if there exists one point (x, t) such that $A_n^{(x,t)} \neq \emptyset$ for all n, that is, if there exists one point with an infinite number of ancestors. Call *clan of ancestors* of (x, t) the union of all its ancestors:

$$A^{(x,t)} = \bigcup_{n \geq 1} A_n^{(x,t)} \tag{2.10}$$

and $R[0, t] = \{R \in \mathbf{R} | \text{ Birth}(R) \in [0, t]\}$.

The existence of the process in infinite volume for any time interval is guaranteed as long as the process does not explode, that is, no rectangle has an infinite number of ancestors in a finite time.

For the existence of the process in infinite time, it is needed that the clan of ancestors of all rectangles are finite with probability one, that is, there is no backward oriented percolation. In order to construct the invariant measure for stationary Markov processes it is usual to construct the process beginning at $-\infty$ with an arbitrary configuration and look at the process at time 0. If the configuration at time 0 does not depend on the initial configuration then we have a sample of invariant measure. The graphical construction described above allow us to construct the process η_t by a thinning of the free process α_t for all $t \in \mathbb{R}$. Moreover, the same argument shows that the distribution of η_0 does not depend on the initial configuration. The above results can be found in Fernández et al. (2001, 2002) and Garcia (2000).

One way of determining the lack of percolation is the domination through a branching process. Establishing sub-criticality conditions for the branching process we obtain sufficient conditions for lack of percolation. Looking backward, the ancestors will be the branches. The time of the death will be the birth time for the branching process. The clan of ancestors in itself is not a branching process because the lack of independence.

3 Dominating the clan of ancestors by a branching process. Critical value.

Let R be a rectangle with basis $\gamma = (x, x + u)$ with length u, born at time 0. Define $\tilde{b}_n^u(v)$ as the number of rectangles in the nth generation of ancestors of R having basis with length v:

$$\tilde{b}_n^u(v) = |\{R' \in A_n^R | \ |\text{Basis}(R')| = v\}|. \tag{3.1}$$

The process $\tilde{b}_n$ is not a Galton-Watson process but it can be dominated by one (call it b_n) as described by Fernández et al. (2001), where each call length represents a type. The number of types can be finite, countable or uncountable depending upon the distribution π.

Lemma 3.2. *The offspring distribution of b_n is Poisson distributed with mean*

$$m(u, v) = \lambda\, \pi(v)\, (u + v) \tag{3.3}$$

where $m(u, v)$ is the mean number of children type v for parents type u.

Proof. In the proof we use the terms "parent" and "ancestor" in the original sense. If $\gamma = (0, u)$ and we consider the rectangle R born at time 0 such that $Basis(R) = \gamma$, it is easy to see that a rectangle $(x, x + v) \times (y, y + s)$ can be a parent of R if, and only if, $x \in (-v, u)$ and $y + s > 0$.

Let $\beta_{uv}(t)$ the number of parents of R type v born after time $-t$. Then

$$b_1^u(v) = \lim_{t\to\infty} \beta_{uv}(t) \quad \text{a.s.} \tag{3.4}$$

Let us call Δ the set $[-v, u] \times [-t, 0]$, and $N(\Delta)$ the homogeneous Poisson process with rate λ in Δ. Then, for $k = 0, 1, \ldots$

$$P(\beta_{uv}(t) = k) =$$
$$\sum_{n\geq k} P(N(\Delta) = n \ \text{ and among } n \text{ rectangles } k \text{ are parents of } R \text{ type } v). \tag{3.5}$$

Let $(x_1, y_1), \ldots, (x_n, y_n)$ a realization of $N(\Delta)$. To each point we associate two independent marks– w, the call length π distributed and s time length exponentially distributed with mean one. Given $N(\Delta) = n$, the points (x_i, y_i) are uniformly distributed in Δ, that is, $x_i \sim U(-v, u)$ and $y_i \sim U(-t, 0)$. Consider the rectangles $R_i = [x_i, x_i + w_i] \times [y_i, y_i + s_i]$. Thus,

$$P(R_i \text{ is a parent of } R \text{ type } v) = \pi(v)\ P(y_i + s_i > 0). \tag{3.6}$$

and we have

$$P(y_i + s_i > 0) = \int_{-t}^{0} P(s_i > -y)\frac{1}{t}dy = \frac{1 - e^{-t}}{t}. \tag{3.7}$$

To clarify the computations we use the following notation:

$$\alpha_t = \lambda\, (u + v)\, t, \qquad p_t = \pi(v)\, (1 - e^{-t})/t.$$

From (3.5),(3.6) and (3.7) we have

$$\mathbb{P}(\beta_{uv}(t)=k)=\sum_{n\geq k}\binom{n}{k}(p_t)^k(1-p_t)^{n-k}e^{-\alpha_t}\frac{(\alpha_t)^n}{n!}=e^{-p_t\alpha_t}\frac{(p_t\alpha_t)^k}{k!}. \quad (3.8)$$

Observe that

$$\lim_{t\to\infty} p_t\alpha_t=\lim_{t\to\infty}\lambda\pi(v)(u+v)(1-e^{-t})=\lambda\pi(v)(u+v). \quad (3.9)$$

From (3.4) it follows that $\beta_{uv}(t)$ converges to $b_1^u(v)$ in distribution

$$\mathbb{P}(b_1^u(v)=k)=\lim_{t\to\infty}P(\beta_{uv}(t)=k),\quad k=0,1,... \quad (3.10)$$

Therefore we conclude that $b_1^u(v)$ has Poisson distribution with mean $\lambda\pi(v)(u+v)$. □

We are interested to find conditions under which the process b_n is sub-critical and a sufficient condition for this is that the mean of the total number of children in all generations when the initial parent is of type u is finite for all u. Thus we are interested in the convergence of the series

$$\sum_{n\geq 1}\sum_{v} m^{(n)}(u,v) \quad (3.11)$$

where $m^{(n)}(u,v)$ is the mean offspring number of type v from a parent type u in the nth generation and it is given inductively by

$$m^{(n)}(u,v)=\sum_{w} m^{(n-1)}(u,w)m(w,v). \quad (3.12)$$

Thus,

$$\sum_{v} m^{(n)}(u,v)=$$
$$\sum_{v}\sum_{v_1}\dots\sum_{v_{n-1}}\lambda^n\pi(v_1)(u+v_1)\pi(v_2)(v_1+v_2)\dots\pi(v)(v_{n-1}+v). \quad (3.13)$$

In order to simplify the reading, recall that ρ_1 and ρ_2 are the first and second moment of the distribution π respectively.

Observe that

$$\begin{aligned}\sum_{v}\pi(v)(v_{n-1}+v)&=v_{n-1}\sum_{v}\pi(v)+\sum_{v}\pi(v)v\\&=v_{n-1}+\rho_1=f_1+v_{n-1}g_1\end{aligned} \quad (3.14)$$

where $f_1 = \rho_1$, $g_1 = 1$. Also,

$$\begin{aligned}
&\sum_{v_{n-1}} \pi(v_{n-1})(v_{n-2} + v_{n-1})(f_1 + v_{n-1}g_1) \\
&= \sum_{v_{n-1}} \pi(v_{n-1})(v_{n-2} + v_{n-1})(v_{n-1} + \rho_1) \\
&= \sum_{v_{n-1}} v_{n-1}^2 \pi(v_{n-1}) + v_{n-1}\pi(v_{n-1})(v_{n-2} + \rho_1) + \pi(v_{n-1})(v_{n-2}\rho_1) \\
&= \rho_2 + \rho_1(v_{n-2} + \rho_1) + v_{n-2}\rho_1 = \rho_2 + \rho_1^2 + v_{n-2}2\rho_1 \\
&= f_2 + v_{n-2}g_2
\end{aligned} \tag{3.15}$$

where $f_2 = \rho_2 + \rho_1^2$, $g_2 = 2\rho_1$.

Analogously,

$$f_{j+1} = \rho_1 f_j + \rho_2 g_j, \quad g_{j+1} = f_j + \rho_1 g_j \tag{3.16}$$

or written in matricial form

$$\begin{bmatrix} f_{j+1} \\ g_{j+1} \end{bmatrix} = T \cdot \begin{bmatrix} f_j \\ g_j \end{bmatrix} = T^j \cdot \begin{bmatrix} \rho_1 \\ 1 \end{bmatrix} \tag{3.17}$$

where

$$T := \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & \rho_1 \end{bmatrix}. \tag{3.18}$$

From (3) it follows

$$\sum_v m^{(n)}(u, v) = \lambda^n (f_n + u\ g_n). \tag{3.19}$$

Computation of f_n and g_n. We need to find T^n, where T is given by (3.18). Its eigenvalues are

$$\epsilon_1 = \rho_1 + \sqrt{\rho_2}, \quad \epsilon_2 = \rho_1 - \sqrt{\rho_2} \tag{3.20}$$

with corresponding right normalized eigenvectors

$$\mathbf{x_1} = \frac{1}{\sqrt{\rho_2 + 1}} \begin{bmatrix} \sqrt{\rho_2} \\ 1 \end{bmatrix}, \quad \mathbf{x_2} = \frac{1}{\sqrt{\rho_2 + 1}} \begin{bmatrix} \sqrt{\rho_2} \\ -1 \end{bmatrix}. \tag{3.21}$$

Then

$$T^n = \frac{1}{2\sqrt{\rho_2}} \begin{bmatrix} \sqrt{\rho_2}(\epsilon_1^n + \epsilon_2^n) & \rho_2(\epsilon_1^n - \epsilon_2^n) \\ \epsilon_1^n - \epsilon_2^n & \sqrt{\rho_2}(\epsilon_1^n + \epsilon_2^n) \end{bmatrix}. \tag{3.22}$$

Now

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = T^{n-1} \cdot \begin{bmatrix} \rho_1 \\ 1 \end{bmatrix} \tag{3.23}$$

giving us

$$f_n = \frac{1}{2}(\epsilon_1^n + \epsilon_2^n), \qquad g_n = \frac{1}{2\sqrt{\rho_2}}(\epsilon_1^n - \epsilon_2^n). \tag{3.24}$$

The radius of convergence λ_c^* of the series

$$\sum_n \sum_v m^{(n)}(u, v) = \sum_n \lambda^n (f_n + u\ g_n) \tag{3.25}$$

is given by the Cauchy-Hadamard formula,

$$\lambda_c^* = \frac{1}{\overline{\lim}_{n\to\infty}(f_n + ug_n)^{1/n}}. \tag{3.26}$$

In order to find λ_c^*, notice

$$f_n + ug_n = \frac{1}{2}\left[\left(1 + \frac{u}{\sqrt{\rho_2}}\right)\epsilon_1^n + \left(1 - \frac{u}{\sqrt{\rho_2}}\right)\epsilon_2^n\right] \tag{3.27}$$

$$= \epsilon_1^n \frac{1}{2}\left[\left(1 + \frac{u}{\sqrt{\rho_2}}\right) + \left(1 - \frac{u}{\sqrt{\rho_2}}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right)^n\right]. \tag{3.28}$$

We know that $\epsilon_1 = \rho_1 + \sqrt{\rho_2}$ is positive since $\pi([0, \infty)) = 1$ and if we had $\rho_1 = 0$ then $\pi(\{0\}) = 1$ which can be excluded. Also ϵ_2 is non-positive, since $\rho_1 \leq \sqrt{\rho_2}$. Moreover, $\epsilon_2 + \epsilon_1 = 2\rho_1 > 0$ and we get $\frac{\epsilon_2}{\epsilon_1} \in [-1, 0]$. Therefore,

$$\begin{aligned} \frac{1}{2}\, 2\min\left(1, \frac{u}{\sqrt{\rho_2}}\right) &\leq \frac{1}{2}\left[\left(1 + \frac{u}{\sqrt{\rho_2}}\right) + \left(1 - \frac{u}{\sqrt{\rho_2}}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right)^n\right] \\ &\leq \frac{1}{2}\, 2\max\left(1, \frac{u}{\sqrt{\rho_2}}\right). \end{aligned} \tag{3.29}$$

From (3.29) and (3.28) we get

$$\epsilon_1\big(\min(1, \frac{u}{\sqrt{\rho_2}})\big)^{1/n} \leq (f_n + ug_n)^{1/n} \leq \epsilon_1\big(\max(1, \frac{u}{\sqrt{\rho_2}})\big)^{1/n}. \tag{3.30}$$

Then,

$$\lim_{n\to\infty}(f_n + ug_n)^{1/n} = \epsilon_1 \tag{3.31}$$

and

$$\lambda_c^* = \frac{1}{\epsilon_1} = \frac{1}{\rho_1 + \sqrt{\rho_2}}. \tag{3.32}$$

Since, $\lambda > 0$, we obtain

1. If $\lambda < \lambda_c^*$ (*cf.* (1.2)) then the series (3.25) converges absolutely and consequently b_n is sub-critical.

2. If $\lambda > \lambda_c^*$ the series (3.25) is divergent and the process b_n can be super-critical.

In the general case, we can have an uncountable number of types. Let V be the set of all possible types and observe that the mean number of offsprings in all generations of a parent type u is given by

$$\sum_{n\geq 1}\int_V m^{(n)}(u, dv) \tag{3.33}$$

where

$$m^{(n)}(u, dv) = \int_V m^{(n-1)}(u, dw)m(w, dv) \tag{3.34}$$

can be obtained inductively. In fact,

$$\int_V m^{(n)}(u, dv) = \int_V\int_V \dots \int_V m(u, dv_1)m(v_1, dv_2)\dots m(v_{n-1}, dv). \tag{3.35}$$

Suppose that the distribution of the length of the calls is absolutely continuous with respect to the Lebesgue measure and call π its density. We can write

$$m(u, dv) = \lambda(u + x)\pi(x)dx. \tag{3.36}$$

Then,

$$\int_V m(v_{n-1}, dv) = \int_0^\infty \lambda(v_{n-1} + x)\pi(x)dx = \lambda(v_{n-1} + \rho_1) \\ = \lambda(f_1 + v_{n-1}g_1) \tag{3.37}$$

and

$$\begin{aligned} &\int_V m(v_{n-2}, dv_{n-1}) \cdot \lambda(f_1 + v_{n-1}g_1) \\ &= \int_0^\infty \lambda(v_{n-2} + x)\pi(x) \cdot \lambda(x + \rho_1)dx \\ &= \lambda^2 \int_0^\infty x^2\pi(x) + x(v_{n-2} + \rho_1)\pi(x) + v_{n-2}\rho_1\pi(x)dx \\ &= \lambda^2(\rho_2 + \rho_1(v_{n-2} + \rho_1) + v_{n-2}\rho_1) \\ &= \lambda^2(\rho_2 + \rho_1^2 + v_{n-2}2\rho_1) = \lambda^2(f_2 + v_{n-2}g_2) \end{aligned} \tag{3.38}$$

where $f_1, g_1, f_2, g_2, \ldots$ are given by (3.17). Therefore, the computation is completely analogous to the discrete case and

$$\int_V m^{(n)}(u, dv) = \lambda^n(f_n + ug_n) \tag{3.39}$$

and the process is sub-critical if the series (3.25) is convergent.

Remark. If $\pi(x) = \mathbf{1}_{[0,1]}(x)$ (the $U(0, 1)$ distribution) then $\lambda_c^* \approx 0.9282$.

Fernández et al. (2002) obtained a sufficient condition for sub-criticality of the branching process which can be written in our case as

$$\alpha = \sup_{(x,u)\in\mathcal{G}} \frac{1}{q((x, u))} \int_{\mathbb{R}} \lambda dy \int_{G_y} \pi(dw)q((y, w))I((y, w), (x, u)) < 1 \tag{3.40}$$

where G_y is the possible set of calls beginning at y and I is defined by (2.5) and q is an arbitrary function such that $q((x, u)) \geq 1$, for all calls (x, u). Due to the translation invariance property of the process, we can consider, without loss of generality, a call beginning at the origin ($x = 0$). Its ancestors would be rectangles, with sufficient long lives, with basis that intersects it. This includes any call beginning at any point inside the call and also all calls beginning before the origin but with sufficient large length to intersect the call. If we choose, $q((x,u)) = c$, where $c \geq 1$ is an arbitrary constant we obtain

$$\alpha \;=\; \lambda \sup_L \left(\int_{-\infty}^{0} P(|H| > -x)dx + \int_0^L dx \right) \;=\; \lambda(\rho_1 + \bar{L}) \quad (3.41)$$

where $\bar{L}$ is defined as $\inf\{L; \pi([0,L]) = 1\}$. This bound coincides with (1.2) only in the case of fixed length call, for all other cases it is weaker than(1.2) since $\sqrt{\rho_2} \leq \bar{L}$. For the particular case, $U(0,1)$, this condition guarantees the sub-criticality of the process for $\lambda < \frac{2}{3} \approx 0.6667$ while our condition gives $\lambda < 0.9282$.

If π does not have bounded support, considering $q((x,u)) = \min(1,u)$ yields the condition

$$\alpha \leq \lambda(\rho_1 + \rho_2 + 1). \quad (3.42)$$

Since our condition is based on the Peron-Frobenius root of $m(u,v)$ then the bound for sub-criticality (λ_c^*) of the branching process is optimal.

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